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On basic forbidden patterns of functions

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ABSTRACT

The allowed patterns of a map on a one-dimensional interval are those permutations that are realized by the relative order of the elements in its orbits. The set of allowed patterns is completely determined by the minimal patterns that are not allowed. These are called basic forbidden patterns.

In this paper, we study basic forbidden patterns of several functions. We show that the logistic map $L_r(x) = rx(1-x)$ and some generalizations have infinitely many of them for $1 < r \leq 4$, and we give a lower bound on the number of basic forbidden patterns of L_4 of each length. Next, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map. Finally, we provide some necessary conditions for a set of permutations to be the set of basic forbidden patterns of such a map.

1. Introduction and definitions

Given a map on a one-dimensional interval, consider the finite sequences (orbits) that are obtained by iterating the map, starting from any point in the interval. The permutations given by the relative order of the elements of these sequences are called *allowed patterns*; permutations that do not appear in this way are called *forbidden patterns*. It was shown in [2,4] that piecewise monotone maps always have forbidden patterns; that is, there are some permutations that do not appear in any orbit. This idea can be used to distinguish random sequences, where every permutation appears with some positive probability, from deterministic sequences produced by iterating a map. Practical aspects of this idea are discussed in [3].

Minimal forbidden patterns, that is, those for which any proper consecutive subpattern is allowed, are called *basic forbidden patterns*. They form an antichain in the partially ordered set of permutations ordered by consecutive pattern containment (see below for definitions), and they contain all the information about the allowed and forbidden patterns of the map.

Consecutive patterns in permutations were first studied in [7] from an enumerative point of view. More recently, they have come up in connection to dynamical systems in [2,4,6].

In this paper, we seek to better understand the set of basic forbidden patterns of functions. Given a map, a natural question is to ask whether its set of basic forbidden patterns is finite or infinite. In Section 2, we give some easy examples of maps with a finite set of basic forbidden patterns. In Section 3, we show that the set of basic forbidden patterns of the logistic map is infinite, and we find some properties of these patterns. We show that the result also holds for a more general class of maps.

Section 4 deals with an important practical question. If we are looking for missing patterns in a sequence in order to tell whether it is random or it has been produced by iterating a piecewise monotone map, it is very useful to have an upper

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bound on the longest patterns whose presence or absence we need to check. In Section 4, we provide an upper bound on the length of the shortest forbidden pattern of a map, based on its number of monotonicity intervals.

Another interesting problem is to characterize what sets of permutations can be the basic forbidden patterns of some piecewise monotone map. In Section 5, we give some necessary conditions that these sets have to satisfy.

1.1. Permutations and consecutive patterns

Denote by $\mathcal{S}_n$ the set of permutations of $\{1, 2, \ldots, n\}$. Let $\mathcal{S} = \cup_{n\geq 1} \mathcal{S}_n$. If $\pi \in \mathcal{S}_n$, we write its one-line notation as $\pi = \pi(1)\pi(2)\cdots\pi(n)$. Sometimes it will be convenient to insert commas between the entries.

Let $x_1, \ldots, x_n \in \mathbb{R}$ with $x_1 < x_2 < \cdots < x_n$. A permutation of $x_1, \ldots, x_n$ can be expressed as $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$, where $\sigma \in \mathcal{S}_n$. We define its *reduction* as

$$\rho(x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}) = \sigma(1)\sigma(2)\cdots\sigma(n) = \sigma.$$

In other words, the reduction is a relabeling of the entries with the numbers $1, 2, \ldots, n$ while preserving the order relationships among them. For example, $\rho(3, 4.2, -2, \sqrt{3}, 1) = 45\,132$.

Given two permutations $\pi \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$ with $m \geq n$, we say that π *contains* σ *as a consecutive pattern* if there exists i such that $\rho(\pi(i)\pi(i+1)\cdots\pi(i+n-1)) = \sigma$. In this case, we also say that σ is a consecutive subpattern of π, and we write $\sigma \preceq \pi$. Otherwise, we say that π *avoids* σ *as a consecutive pattern*. In the rest of the paper, all the notions of pattern containment and avoidance refer to the consecutive case, even if the word *consecutive* is omitted. Denote by $\mathrm{Av}_n(\sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid σ as a consecutive pattern, and let $\mathrm{Av}(\sigma) = \cup_{n\geq 1}\mathrm{Av}_n(\sigma)$. In general, if $\Sigma \subset \mathcal{S}$, let $\mathrm{Av}(\Sigma)$ be the set of permutations that avoid all the patterns in Σ, and let $\mathrm{Av}_n(\Sigma) = \mathrm{Av}(\Sigma) \cap \mathcal{S}_n$. Consecutive pattern containment (and avoidance) was first studied in [7]. In [5], the asymptotic behavior of the number of permutations that avoid a consecutive pattern π is considered.

Theorem 1.1 (*[5]*). *Let* $\sigma \in \mathcal{S}_k$ *with* $k \geq 3$. *Then there exist constants* $0 < c, d < 1$ *such that*

$$c^n n! < |\mathrm{Av}_n(\sigma)| < d^n n!$$

for all $n \geq k$.

The consecutive containment relation $\preceq$ defines a partial order on $\mathcal{S}$. Denote by $P_c = (\mathcal{S}, \preceq)$ the resulting infinite partially ordered set. We say that a set $A \subset \mathcal{S}$ is a *closed consecutive permutation class* if it is closed under consecutive pattern containment, that is, if $\pi \in A$ and $\sigma \preceq \pi$ imply that $\sigma \in A$. In this case, the *basis* of A consists of the minimal permutations not in A; that is,

$$\mathrm{Bas}(A) = \{\pi \in \mathcal{S} \setminus A : \text{if } \sigma \preceq \pi, \sigma \neq \pi \text{ then } \sigma \in A\}.$$

Note that $\mathrm{Bas}(A)$ is an antichain in P_c; that is, there are no two permutations $\tau, \pi \in \mathrm{Bas}(A)$ with $\tau \neq \pi$ and $\tau \preceq \pi$. Conversely, any antichain Σ is the basis of the closed class $\mathrm{Av}(\Sigma)$. This gives a one-to-one correspondence between antichains of P_c and closed consecutive permutation classes.

For example, if A is the set of up–down or down–up permutations, i.e., those permutations satisfying $\pi(1) < \pi(2) > \pi(3) < \pi(4) > \cdots$ or $\pi(1) > \pi(2) < \pi(3) > \pi(4) < \cdots$, then $\mathrm{Bas}(A) = \{123, 321\}$. If B is the antichain $\{132, 231\}$, then $\mathrm{Av}(B)$ is the set of permutations having no *peaks*, i.e., no i such that $\pi(i-1) < \pi(i) > \pi(i+1)$.

1.2. Allowed and forbidden patterns of maps

Let $f : I \to I$, where $I \subset \mathbb{R}$ is a closed interval. Given $x \in I$ and $n \geq 1$, let

$$\mathrm{Pat}(x, f, n) = \rho(x, f(x), f^2(x), \ldots, f^{n-1}(x)),$$

provided that there is no pair $0 \leq i < j \leq n-1$ such that $f^i(x) \neq f^j(x)$. If such a pair exists, then $\mathrm{Pat}(x, f, n)$ is not defined. When it is defined, $\mathrm{Pat}(x, f, n) \in \mathcal{S}_n$. For example, if $L_4 : [0, 1] \to [0, 1]$ is the *logistic map* $L_4(x) = 4x(1-x)$ and we take $x = 0.8$ to be the initial value, then

$$(x, L_4(x), L_4^2(x), L_4^3(x)) = (0.8, 0.64, 0.9216, 0.28901376),$$

so $\mathrm{Pat}(0.8, L_4, 4) = 3241$.

If $\pi \in \mathcal{S}_n$ and there is some $x \in I$ such that $\mathrm{Pat}(x, f, n) = \pi$, we say that π is *realized* by f (at x), or that π is an *allowed pattern* of f. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f) = \bigcup_{n\geq 1}\mathrm{Allow}_n(f)$, where

$$\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n.$$

The remaining permutations are called *forbidden patterns*, and are denoted by $\mathrm{Forb}(f) = \mathcal{S} \setminus \mathrm{Allow}(f)$.

It is noticed in [6] that $\mathrm{Allow}(f)$ is closed under consecutive pattern containment: if $\mathrm{Pat}(x, f, n) = \pi$ and $\tau \preceq \pi$, then there exist i, j such that $\rho(\pi(i)\pi(i+1)\cdots\pi(j)) = \tau$; hence $\mathrm{Pat}(f^{i-1}(x), f, j-i+1) = \tau$. Those forbidden patterns for which any proper subpattern is allowed are called the *basic forbidden patterns* of f, and are denoted $\mathrm{B}(f)$. This set is an antichain and it is the basis of Allow(f); i.e., $\mathrm{B}(f) = \mathrm{Bas}(\mathrm{Allow}(f))$. In particular, we have that $\mathrm{Allow}(f) = \mathrm{Av}(\mathrm{B}(f))$. We will use the notation $\mathrm{B}_n(f) = \mathrm{B}(f) \cap \mathcal{S}_n$.

For example, if $g : [0, 1] \to [0, 1]$ is the map $g(x) = 1 - x^2$, then $\mathrm{B}(g) = \{123, 132, 312, 321\}$. To see this, note that the graphs of $x, g(x), g^2(x), \dots$ all intersect at the point $\left(\frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}\right)$, and that

$$\cdots < g^6(x) < g^4(x) < g^2(x) < x < g(x) < g^3(x) < g^5(x) < \cdots$$

for $0 < x < \frac{\sqrt{5}+1}{2}$ and

$$\cdots > g^6(x) > g^4(x) > g^2(x) > x > g(x) > g^3(x) > g^5(x) > \cdots$$

for $\frac{\sqrt{5}+1}{2} < x < 1$. Other simple cases where the set of basic forbidden patterns is finite and easy to compute are discussed in Section 2.

For many maps, however, the set of basic forbidden patterns is infinite. For the map L_4 defined above, it can be checked that $\mathrm{B}_3(L_4) = \{321\}$ and

$$\mathrm{B}_4(L_4) = \{1423, 2134, 2143, 3142, 4231\}.$$

In Section 3, we study the set of basic forbidden patterns of the logistic map

$$\begin{array}{rlcl} L_r : & [0, 1] & \longrightarrow & [0, 1] \\ & x & \mapsto & rx(1-x), \end{array}$$

where $1 < r \leq 4$. We show that $|\mathrm{B}(L_r)|$ is infinite for $1 < r \leq 4$, and that $|\mathrm{B}_n(L_4)| \geq n - 1$. We prove that some generalizations of these maps also have infinitely many forbidden patterns.

The following important result, which in implicit in [4], guarantees that, under mild conditions on f, the set $\mathrm{B}(f)$ is nonempty. Recall that piecewise monotone means that there exists a finite partition of I into intervals where f is continuous and strictly monotone.

Proposition 1.2. *If $f : I \to I$ is piecewise monotone, then* $\mathrm{Forb}(f) \neq \emptyset$. *In particular,* $\mathrm{B}(f) \neq \emptyset$.

In fact, it is shown in [4] that, for such a map, $\lim_{n\to\infty} \frac{1}{n} \log |\mathrm{Allow}_n(f)|$ exists and equals the topological entropy of f, a constant which measures the complexity of the dynamical system. In particular, the number of allowed patterns of f grows at most exponentially; i.e.,

$$|\mathrm{Allow}_n(f)| < C^n \tag{1}$$

for some constant C. Since the total number of permutations of length n grows super-exponentially, the above proposition holds. In fact, as n approaches infinity, most permutations in $\mathcal{S}_n$ are forbidden. In contrast, in a random sequence, all permutations occur with positive probability. Because of this, forbidden patterns can be used to distinguish random time series from deterministic ones, as studied in [3].

It is shown in [2] that there exist non-piecewise monotone maps that realize all permutations in $\mathcal{S}$. Unless otherwise stated, all the maps f in the rest of the paper will be assumed to be piecewise monotone maps on an interval $I \subset \mathbb{R}$.

From a practical perspective, the downside of Proposition 1.2 is that it does not give information about how long the permutations in Forb(f) are. Knowing the length of the shortest forbidden pattern of certain classes of maps is useful when we are trying to distinguish random sequences from chaotic ones generated by orbits of maps in the class. In Section 4, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map.

Another problem that arises when studying forbidden patterns is the characterization of antichains Σ for which there exists a piecewise monotone map f such that $\mathrm{B}(f) = \Sigma$. This is equivalent to asking whether $\mathrm{Av}(\Sigma)$ is the set of allowed patterns of a map, that is, if $\mathrm{Av}(\Sigma) = \mathrm{Allow}(f)$ for some f. It is clear from Eq. (1) that a *necessary condition* on Σ is that $|\mathrm{Av}_n(\Sigma)| < C^n$ for some constant C. For example, if $\Sigma = \{\sigma\}$, where σ has length at least 3, then this condition implies that there is no f such that $\mathrm{Av}(\sigma) = \mathrm{Allow}(f)$. Indeed, by Theorem 1.1, $|\mathrm{Av}_n(\sigma)| > c^n n!$ for some $0 < c < 1$, and this lower bound is larger than the necessary exponential growth. In Section 5, we show that this is not the only necessary condition on the antichain Σ; that is, there are antichains for which $|\mathrm{Av}_n(\Sigma)|$ grows exponentially, yet there is no piecewise monotone map f with $\mathrm{B}(f) = \Sigma$.

2. Functions with known forbidden patterns

Determining the forbidden patterns of an arbitrary map is a wide-open problem. Only a few results are known for specific maps. Some work has been done in [2,6] for the so-called *one-sided shift maps*, or simply *shifts* for short. From a forbidden pattern perspective, the shift on N symbols is equivalent to the *sawtooth map*

$$\begin{array}{rlcl} \mathrm{Saw}_N : & [0, 1] & \longrightarrow & [0, 1] \\ & x & \mapsto & Nx \bmod 1, \end{array}$$

as shown in [2].

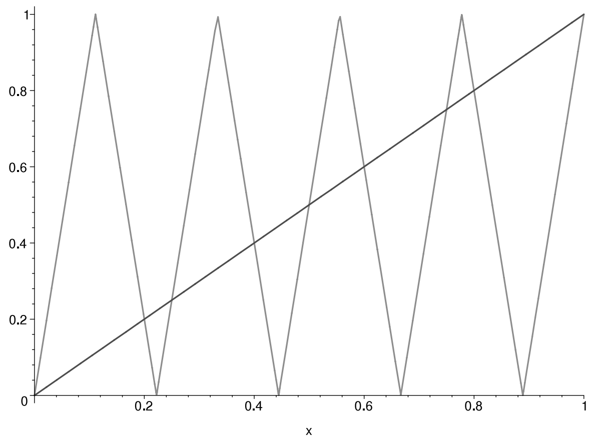

Fig. 1. The alternating sawtooth map Alt_9.

It is proved in [2] that shifts (equivalently, sawtooth maps) have infinitely many basic forbidden patterns. A characterization of forbidden patterns of these maps is given in [6], providing a formula to compute, for a given permutation π, the smallest N such that π is realized by Saw_N. The sets $\mathrm{Allow}_n(\mathrm{Saw}_N)$ are enumerated for all n and N. To our knowledge, shifts are the only non-trivial maps for which forbidden patterns have been characterized.

A generalization of shifts is the so-called *signed shifts*, which are equivalent to *signed sawtooth maps*. Roughly speaking, for each one of the N spikes of slope N in the graph of Saw_N, one can choose to replace it with a spike of slope $-N$. For example, for $N = 2$, if we reverse the second spike, we obtain the *tent map* $\Lambda : [0, 1] \to [0, 1]$ defined by

$$\Lambda(x) = \begin{cases} 2x & \text{if } 0 \le x < \dfrac{1}{2}, \\ 2 - 2x & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases}$$

A signed sawtooth map where the slopes alternate between positive and negative is called an *alternating signed sawtooth map*. Let $\mathrm{Alt}_N : [0, 1] \to [0, 1]$ denote the alternating signed sawtooth map with N ramps, defined by

$$\mathrm{Alt}_N(x) = \Lambda\left(\frac{Nx}{2} \bmod 1\right).$$

The graph of Alt_9 is shown in Fig. 1. Forbidden patterns of signed shifts have recently been studied in [1].

On the other hand, for certain well-behaved functions, the description of their allowed and forbidden patterns is relatively straightforward.

Lemma 2.1. *Let $f : [0, 1] \to [0, 1]$ be a monotone increasing, continuous function with at least one fixed point on $(0, 1)$. Assume that f is not the identity function.*

- *If $f(x) \ge x$ for all x, then* $\mathrm{B}(f) = \{21\}$;
- *if $f(x) \le x$ for all x, then* $\mathrm{B}(f) = \{12\}$;
- *otherwise,* $\mathrm{B}(f) = \{132, 213, 231, 312\}$.

Proof. Let $U = \{x \in [0, 1] : x < f(x)\}$, $V = \{x \in [0, 1] : x > f(x)\}$. Then U can be decomposed as a union of open intervals (a, b), where a and b are fixed points of f, and possibly an interval $[0, b)$. Since f is increasing and continuous, $f((a, b)) = (a, b)$, and $f([0, b)) \subseteq [0, b)$. Thus, if $x \in U$, then $f(x) \in U$, so $x < f(x) < f^2(x) < \cdots$ for any $x \in U$, and similarly $x > f(x) > f^2(x) > \cdots$ for any $x \in V$.

If $f(x) \ge x$ for all x, then $V = \emptyset$, and the only allowed pattern of length n is $12 \cdots n$, so $\mathrm{B}(f) = \{21\}$. Similarly, if $f(x) \le x$ for all x, then $U = \emptyset$, and the only allowed pattern of length n is $n(n-1)\cdots 1$, so $\mathrm{B}(f) = \{12\}$. In all other cases, $12 \cdots n$ and $n(n-1)\cdots 1$ are the allowed patterns of length n, so $\mathrm{B}(f) = \{132, 213, 231, 312\}$. $\square$

3. The logistic map and generalizations

3.1. Basic forbidden patterns of the logistic map

In this section, we study the basic forbidden patterns of the logistic map

$$\begin{array}{rccc} L_r : & [0, 1] & \longrightarrow & [0, 1] \\ & x & \mapsto & rx(1-x), \end{array}$$

where $1 < r \le 4$. To simplify the notation, we will write L instead of L_4.

It is shown in [2] that L is order-isomorphic to the tent map Λ, and therefore L and Λ have the same allowed and forbidden patterns. It has recently been proved [1] that Λ has infinitely many forbidden patterns, by interpreting it as a signed shift. Here, we generalize this result in two ways. First, we show that all maps L_r with $1 < r \leq 4$ have infinitely many forbidden patterns. Next, we give a linear lower bound on the number of basic forbidden patterns of L of each length.

Proposition 3.1. *For every* $1 < r \leq 4$, $|\mathrm{B}(L_r)|$ *is infinite.*

Proof. Recall that a permutation of length n is a basic forbidden pattern if and only if it is forbidden and its two subpatterns of length $n-1$ are allowed.

First, we show that, for $n \geq 4$, $(n-2)12\cdots(n-3)(n-1)n \in \mathrm{Forb}(L_r)$. Let $a_r = 1 - 1/r$ be the unique fixed point of L_r in $(0, 1)$. It is clear that $\{x \in [0, 1] : x < L_r(x)\} = (0, a_r)$ and $\{x \in [0, 1] : x > L_r(x)\} = (a_r, 1]$. Suppose, contrary to our claim, that there exists some $x \in [0, 1]$ such that $\mathrm{Pat}(x, L_r, n) = (n-2)123\cdots(n-3)(n-1)n$. In other words,

$$L_r(x) < L_r^2(x) < \cdots < L_r^{n-3}(x) < x < L_r^{n-2}(x) < L_r^{n-1}(x).$$

Then, for each $1 \leq i \leq n-2$, , $L_r^i(x) \in (0, a_r)$, whereas $x \in (a_r, 1)$. However, since $L_r^{n-3}(x), L_r^{n-2}(x) \in (0, a_r)$, which is an interval, and $L_r^{n-3}(x) < x < L_r^{n-2}(x)$, we must have $x \in (0, a_r)$, leading to a contradiction. Therefore, $(n-2)123\cdots(n-3)(n-1)n \in \mathrm{Forb}(L_r)$.

Next, we show that $12\cdots m \in \mathrm{Allow}(L_r)$ for all m. Since $L_r(x) = rx(1-x) < 4x$ for $x > 0$, we have that $L_r^i(x) < a_r/4^{m-2-i}$ for $x \in (0, a_r/4^{m-2})$ and $0 \leq i \leq m-2$. Now, using that $y < L_r(y)$ for $y \in (0, a_r)$, we get

$$x < L_r(x) < L_r^2(x) < \cdots < L_r^{m-1}(x)$$

for $x \in (0, a_r/4^{m-2})$, so $123\cdots m \in \mathrm{Allow}(L_r)$.

Now we show that $m12\cdots(m-1) \in \mathrm{Allow}(L_r)$ for all m. Let $y \in (1-a_r/4^{m-1}, 1)$ with $y > 3/4$. Then $1-y \in (0, a_r/4^{m-1})$, so, by the same argument as above,

$$1-y < L_r(1-y) = L_r(y) < L_r^2(y) < \cdots < L_r^m(y).$$

Also, $L_r^{m-1}(y) < a_r = \frac{r-1}{r} \leq 3/4 < y$. Thus $\mathrm{Pat}(y, L_r, m) = m12\cdots(m-1)$.

Summarizing, we have shown that $(n-2)12\cdots(n-3)(n-1)n$ is forbidden, but both the subpattern formed by its first $n-2$ entries and the one formed by its last $n-1$ entries are allowed. Now, there are two possibilities. If $(n-2)12\cdots(n-3)(n-1)$ is forbidden, then it must be a basic forbidden pattern. If it is allowed, then $(n-2)12\cdots(n-3)(n-1)n \in \mathrm{B}(L_r)$. Either way, $\mathrm{B}(L_r)$ contains a permutation of length $n-1$ or n, for all $n \geq 4$, so it is an infinite set. □

At the end of the proof above we encountered two possibilities, depending on whether $(n-2)12\cdots(n-3)(n-1)$ is forbidden or allowed. We now show that for $r < 4$ this pattern is forbidden for n large enough, but for $r = 4$ it is allowed, so $(n-2)12\cdots(n-3)(n-1)n \in \mathrm{B}(L)$.

Proposition 3.2. *For each* $1 < r < 4$, *there is some* n_0 *such that, for every* $n \geq n_0$,

$$(n-1)123\cdots(n-2)n \in \mathrm{B}(L_r).$$

Proof. We have shown in the proof of Proposition 3.1 that $(n-1)12\cdots(n-2), 123\cdots(n-1) \in \mathrm{Allow}(L_r)$ for all n. So, it suffices to show that $(n-1)123\cdots(n-2)n \in \mathrm{Forb}(L_r)$ for large enough n. Let $a_r = 1 - 1/r$, as above.

Let $n \geq 3$, and suppose that there exists x such that $\mathrm{Pat}(x, L_r, n) = (n-1)123\cdots(n-2)n$; that is,

$$L_r(x) < L_r^2(x) < \cdots < L_r^{n-2}(x) < x < L_r^{n-1}(x). \tag{2}$$

Since $x > L_r(x)$, we have $x > a_r$. Since $L_r(x) < L_r^2(x)$, we have $L_r(x) < a_r$, and so $x > \max\{a_r, 1-a_r\} = \max\{1-1/r, 1/r\}$.

If $r \leq 2$, then we have $x > 1/r \geq 1/2$. On the other hand,

$$\max_{0 \leq y \leq 1} L_r(y) = \frac{r}{4} \leq \frac{1}{2}.$$

Thus,

$$x > \max_{0 \leq y \leq 1} L_r(y) \geq L_r^{n-1}(x),$$

contradicting (2).

From now on, we assume that $r > 2$. In this case, $x > 1 - 1/r = a_r$. Moreover, for $1 \leq i \leq n-3$, we have from (2) that $L_r^{i+1}(x) < L_r^{i+2}(x)$, so $L_r^{i+1}(x) < a_r$, which implies in turn that $L_r^i(x) < 1 - a_r = 1/r$. Since $L_r(1/r) = 1 - 1/r > 1/r$, there is some $\alpha > 1$ such that

$$L_r\left(\frac{1}{\alpha r}\right) = \frac{1}{\alpha}\left(1 - \frac{1}{\alpha r}\right) = \frac{1}{r}.$$

Now, the fact that

$$L_r^{n-3}(x) < \frac{1}{r} = L_r\left(\frac{1}{\alpha r}\right)$$

implies that $L_r^{n-4}(x) < \frac{1}{\alpha r}$ or $L_r^{n-4}(x) > 1 - \frac{1}{\alpha r}$. Since $L_r^{n-4}(x) < 1/r$, it is the first inequality that holds. By induction on j (we have just done the case $j = 1$), we see that

$$L_r^{n-3-j}(x) < \frac{1}{\alpha^j r} = \frac{1}{\alpha^{j+1}}\left(1 - \frac{1}{\alpha r}\right) \leq \frac{1}{\alpha^{j+1}}\left(1 - \frac{1}{\alpha^{j+1} r}\right) = L_r\left(\frac{1}{\alpha^{j+1} r}\right), \tag{3}$$

and so, for $j \leq n-5$,

$$L_r^{n-4-j}(x) < \frac{1}{\alpha^{j+1} r}.$$

Eq. (3) also holds for $j = n-4$, and from it we get that

$$x < \frac{1}{\alpha^{n-3} r} \quad \text{or} \quad x > 1 - \frac{1}{\alpha^{n-3} r}. \tag{4}$$

Since we know that $x > 1 - 1/r$, the second inequality in (4) must hold. But since $\alpha > 1$ and $r < 4$, there must be an n_0 such that

$$1 - \frac{1}{\alpha^{n_0-3} r} > \frac{r}{4}.$$

Now, for $n \geq n_0$,

$$x > 1 - \frac{1}{\alpha^{n-3} r} > \frac{r}{4} = \max_{0 \leq y \leq 1} L_r(y) \geq L_r^{n-1}(x),$$

again contradicting (2). □

For $r = 4$, the forbidden patterns behave in a different way. The pattern mentioned in Proposition 3.2 is now allowed, but instead we have other basic forbidden patterns. It is shown in [1, Theorem 4.4] that $(n-3)(n-2)(n-1)12\cdots(n-4)n \in \mathrm{B}_n(L)$ for $n \geq 5$. We find here $n-1$ additional basic forbidden patterns of length n, thus giving a linear lower bound on $|\mathrm{B}_n(L)|$.

Proposition 3.3. *For $n \geq 4$, the set $\mathrm{B}_n(L)$ contains the following patterns:*

$$(n-2)12\cdots(n-3)(n-1)n,$$
$$(n-2)12\cdots(n-3)n(n-1),$$
$$(n-1)12\cdots(k-1)(k+1)\cdots(n-2)nk \quad \text{for } 2 \leq k \leq n-2.$$

In particular,

$$|\mathrm{B}_n(L)| \geq n.$$

Proof. It is shown in [2] that $\mathrm{B}(L) = \mathrm{B}(\Lambda)$, so it suffices to prove the statement for the tent map Λ.

First, we show that $(m-1)123\cdots(m-2)m \in \mathrm{Allow}(\Lambda)$. Let

$$x = 1 - \frac{1+2^{-m}}{2^{m-1}+1}.$$

Then $\Lambda(x) = 2(1-x)$. In general, for $1 \leq i \leq m-2$,

$$\Lambda^i(x) = 2^i(1-x) = \frac{2^i + 2^{i-m}}{2^{m-1}+1} \leq \frac{2^{m-2}+1/4}{2^{m-1}+1} < \frac{1}{2} < x,$$

and

$$\Lambda^{m-1}(x) = 2^{m-1}(1-x) = \frac{2^{m-1}+1/2}{2^{m-1}+1} > x.$$

Thus,

$$\Lambda(x) < \Lambda^2(x) < \cdots < \Lambda^{m-2}(x) < x < \Lambda^{m-1}(x),$$

so $\mathrm{Pat}(x, \Lambda, m) = (m-1)123\cdots(m-2)m$.

Now, we show that $123\cdots(k-1)(k+1)\cdots(m-1)mk \in \mathrm{Allow}(\Lambda)$ for $1 \le k \le m$. Since

$$\{x \in [0,1] : x < \Lambda(x)\} = (0, 2/3)$$

and

$$\{x \in [0,1] : \Lambda(x) < c\} = (0, c/2) \cup (1 - c/2, 1)$$

for any $0 < c < 1$, we have that

$$\{x \in [0,1] : x < \Lambda(x) < \Lambda^2(x)\} = (0, 1/3).$$

In general, it is easy to see that, for $m \ge 3$,

$$\{x \in [0,1] : x < \Lambda(x) < \Lambda^2(x) < \cdots < \Lambda^{m-2}(x)\} = \left(0, \frac{1}{3 \cdot 2^{m-4}}\right). \tag{5}$$

Since $\Lambda^{m-1}(x)$ is continuous, $\Lambda^{m-1}(1/2^{m-1}) = 1$, and $\Lambda^{m-1}(1/2^{m-2}) = 0$, the graph of $\Lambda^{m-1}(x)$ intersects the graph of $\Lambda^i(x)$, for each $0 \le i \le m-2$, in the interval

$$\left[\frac{1}{2^{m-1}}, \frac{1}{2^{m-2}}\right] \subset \left(0, \frac{1}{3 \cdot 2^{m-4}}\right),$$

thus realizing the patterns $123\cdots(k-1)(k+1)\cdots mk$ for each $1 \le k \le m$.

Since for each of the patterns in the statement of the proposition both of its subpatterns of length $n-1$ are allowed, it suffices to show that they are forbidden to conclude that they are in $\mathrm{B}(f)$.

Suppose that there exists $x \in [0,1]$ such that $\mathrm{Pat}(x, \Lambda, n)$ is one of the listed patterns. Then

$$\Lambda(x) < \Lambda^2(x) < \cdots < \Lambda^{n-3}(x) < x < \Lambda^{n-2}(x).$$

Since $x > \Lambda(x)$, we have that $x > 2/3$. And since $\Lambda^{n-2}(x) > x > 1/2$, we have that

$$\Lambda^{n-1}(x) = 2(1 - \Lambda^{n-2}(x)) < 2(1 - x) = \Lambda(x).$$

Therefore,

$$\Lambda^{n-1}(x) < \Lambda(x) < \Lambda^2(x) < \cdots < \Lambda^{n-3}(x) < x < \Lambda^{n-2}(x),$$

so $\mathrm{Pat}(x, \Lambda, n) = (n-1)23\cdots(n-2)n1$, and all the patterns in the statement are forbidden.

Together with the fact that $(n-3)(n-2)(n-1)12\cdots(n-4)n \in \mathrm{B}_n(L)$ for $n \ge 5$ and that $|\mathrm{B}_4(L)| = 5$, the lower bound $|\mathrm{B}_n(L)| \ge n$ follows. $\square$

In fact, we expect the actual size of $\mathrm{B}_n(L)$ to grow much faster than this. For $n \ge 3$, the first few values of $|\mathrm{B}_n(L)|$, found by computer, are 1, 5, 9, 28, 53, 110, ...

3.2. Generalizations

It is an interesting open problem to characterize those maps for which the set of basic forbidden patterns in infinite. We showed that this is the case for L_r. Here, we give a sufficient set of conditions on f that makes $\mathrm{B}(f)$ infinite. The conditions generalize some properties of the logistic map that we used in the subsection above, including symmetry, and having a single fixed point in $(0, 1)$.

Proposition 3.4. *Let $f : [0,1] \to [0,1]$ be a continuous function that satisfies the following conditions:*

1. $f(0) = 0$,
2. $f(x) = f(1-x)$ *for all* x,
3. $f(x)$ *has a single fixed point in* $(0, 1)$,
4. *there is some x such that* $f(x) > x$.

Then f has infinitely many basic forbidden patterns.

Note that in the proposition above we do not require f to be piecewise monotone.

Proof. This proof is similar to that of Proposition 3.1. For each $i \ge 1$, let

$$T_i = \{x \in [0,1] : x < f(x) < f^2(x) < \cdots < f^i(x)\}.$$

Conditions 1, 3, and 4 imply that $T_1 = (0, t_1)$, where t_1 is the fixed point of f in $(0, 1)$.

We claim that there exists a decreasing sequence of positive numbers $t_1 > t_2 > t_3 > \cdots$ such that $(0, t_i) \subseteq T_i$ for each $i \ge 1$. We show that t_i exists by induction on i. Assume that $i \ge 2$ and that we have shown the existence of t_{i-1} with $(0, t_{i-1}) \subseteq T_{i-1}$. Let t_i be the smallest root of $f(x) = t_{i-1}$. Clearly, $t_i \le t_{i-1}$. Let $x \in (0, t_i)$. Since $f(0) = 0 < t_{i-1}$, we have that $f(x) < t_{i-1}$, so $f(x) \in T_{i-1}$. By the definition of $T_{i-1}, f(x) < f^2(x) < \cdots < f^i(x)$. Moreover, since $x < t_i \leq t_{i-1}, x \in T_{i-1}$, so $x < f(x)$. It follows that $x \in T_i$. Hence $(0, t_i) \subseteq T_i$. This proves that, for all m, $T_{m-1} \neq \emptyset$, so $12\cdots m \in \text{Allow}(f)$.

We next show that, for $m \geq 3$, $m123\cdots(m-1) \in \text{Allow}(f)$. Let $x \in (\max\{t_1, 1-t_m\}, 1)$. Then $1-x \in T_m$, so

$$1-x < f(1-x) = f(x) < f^2(x) < \cdots < f^m(x).$$

Since $f^{m-1}(x) < f^m(x)$, we have that $f^{m-1}(x) \in T_1$, so $f^{m-1}(x) < t_1 < x$. Thus

$$f(x) < f^2(x) < \cdots < f^{m-1}(x) < x,$$

which shows that $m123\cdots(m-1) \in \text{Allow}(f)$.

Now we prove that, for $n \geq 4$, $(n-2)12\cdots(n-3)(n-1)n \in \text{Forb}(f)$. Suppose that there exists some $x \in [0, 1]$ such that $\text{Pat}(x, f, n) = (n-2)12\cdots(n-3)(n-1)n$. Then $x > f(x)$ and $f^{n-3}(x) < f^{n-2}(x) < f^{n-1}(x)$. Therefore, $x \notin T_1$, whereas $f^{n-3}(x), f^{n-2}(x) \in T_1$. However, because $T_1 = (0, t_1)$ is an interval and $f^{n-3}(x) < x < f^{n-2}(x)$, we must have $x \in T_1$, which is a contradiction.

Summarizing, we have shown that $(n-2)12\cdots(n-3)(n-1)n$ is forbidden, but both the subpattern formed by its first $n-2$ entries and the one formed by its last $n-1$ entries are allowed. If $(n-2)12\cdots(n-3)(n-1)$ is forbidden, then it must be a basic forbidden pattern; if it is allowed, then $(n-2)12\cdots(n-3)(n-1)n \in \text{B}(f)$. Either way, $\text{B}(f)$ contains a permutation of length $n-1$ or n, for all $n \geq 4$. □

Obviously, the conditions in Proposition 3.4 are not necessary for a map to have infinitely many forbidden patterns. For example, as mentioned above, it is known [2] that the maps Saw_N for $N \geq 2$ have infinitely many forbidden patterns.

4. The shortest forbidden pattern

In this section, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map $f : I \to I$. The intervals in the partition of I such that f is continuous and strictly monotone on each interval are called the *monotonicity intervals* of f.

Theorem 4.1. *Let $f : I \to I$ be a piecewise monotone map with m monotonicity intervals, and let $k \leq m$ be the number of such intervals I' with one of these two properties:*

- *f is increasing in I' and the left endpoint a of $\overline{I'}$ satisfies $f(a) < a$,*
- *f is decreasing in I' and I' contains a point with $f(x) < x$.*

Then the length of the shortest forbidden pattern of f is at most $2k+3$.

Proof. Let $I = \bigcup_{j=1}^{m} I_j$ be the finite partition into intervals where f is continuous and strictly monotone. Let

$$D = \{x \in I : f(x) < x\} = \bigcup_{\alpha} D_\alpha$$

expressed as the union of its connected components, where each D_α is an interval (which can consist of a single point). Note that this can be an infinite union.

Let $1 \leq j \leq m$. If f is decreasing in I_j, then I_j can intersect at most one of the D_α. Suppose now that f is increasing in I_j, and let $\overline{I_j} = [a_j, b_j]$ and $\overline{D_\alpha} = [c_\alpha, d_\alpha]$. We claim that, if $a_j < c_\alpha < b_j$, then $f(D_\alpha) \subseteq D_\alpha$. Indeed, since f is continuous in I_j and c_α is the left endpoint of D_α, we have that $f(c_\alpha) = c_\alpha$, and so

$$c_\alpha < f(x) < x \leq d_\alpha$$

for all $x \in D_\alpha$, which implies that $f(D_\alpha) \subseteq D_\alpha$. For the same reason, if $a_j = c_\alpha$ but $f(a_j) = a_j$, we have again that $f(D_\alpha) \subseteq D_\alpha$. Therefore, the number of intervals D_α for which $f(D_\alpha) \not\subseteq D_\alpha$ is at most k.

For $n \geq 2k+3$, let $E_n \subset \mathcal{S}_n$ be the set of permutations π for which there exists an i such that

1. for each $i \leq j \leq i+k-1$, there exists $1 \leq \ell \leq n-1$ such that $\pi(j) > \pi(\ell) > \pi(j+1)$ and $\pi(\ell) < \pi(\ell+1)$ (note that this condition implies that $\pi(j) \geq \pi(j+1)+2$),
2. $\pi(i+k) > \pi(i+k+1)$,
3. there is some $h > i+k+1$ such that $\pi(i+k+1) < \pi(h)$.

For example, $\pi = (2k+2)(2k)(2k-2)\cdots 42\,135\cdots(2k+1)(2k+3) \in E_{2k+3}$. We claim that $E_n \subset \text{Forb}(f)$. Once we prove this claim, the theorem will follow from this example.

Suppose that for some $\sigma \in E_n$ there is a $y \in I$ such that $\text{Pat}(y, f, n) = \sigma$. Then, whenever $\pi(j) > \pi(j+1)$, we have $f^{j-1}(y) > f^j(y)$, and thus $f^{j-1}(y) \in D$, and whenever $\pi(\ell) < \pi(\ell+1)$, we have $f^{\ell-1}(y) < f^\ell(y)$, and thus $f^{\ell-1}(y) \notin D$.

For each $i \leq j \leq i+k$, we have $f^{j-1}(y) \in D_{\alpha_j}$ for some α_j. If $f(D_{\alpha_j}) \subseteq D_{\alpha_j}$, then $f^{j-1}(y) > f^j(y) > f^{j+1}(y) > \cdots$. But this is impossible, because condition 3 implies that $f^{i+k}(y) < f^{h-1}(y)$ for some $h > i+k+1$. Therefore, $f(D_{\alpha_j}) \not\subseteq D_{\alpha_j}$. Now, there are $k+1$ choices for j, and only k different indices α such that $f(D_\alpha) \not\subseteq D_\alpha$. Hence, there must be two different indices $i \leq j < j' \leq i+k$ for which $\alpha_j = \alpha_{j'}$. Since D_{α_j} is an interval, for any $f^{j-1}(y) > z > f^{j'-1}(y)$ we must have $z \in D_{\alpha_j} \subseteq D$. However, by condition 1, there exists ℓ such that $\pi(j) > \pi(\ell) > \pi(j+1) \geq \pi(j')$ and $\pi(\ell) < \pi(\ell+1)$. This implies that $f^{j-1}(y) > f^{\ell-1}(y) > f^{j'-1}(y)$, but at the same time $f^{\ell-1}(y) < f^\ell(y)$, so $f^{\ell-1}(y) \notin D$, which is a contradiction. □

The permutation π in the above proof is not the only element of E_{2k+3}. For example, for $k = 1$, we have $34\,215, 35\,214, 42\,135, 45\,213, 45\,312, 52\,134 \in E_5$.

In many cases, it is more practical to work with the following simplified version of Theorem 4.1.

Theorem 4.2. *Let $f : I \to I$ be a piecewise monotone map and let $D = \{x \in I : f(x) < x\}$. Let k be the number of connected components of D. Then the length of the shortest forbidden pattern of f is at most $2k + 2$.*

Proof. This proof is analogous to that of Theorem 4.1. In this case, the statement already gives the bound k on the number of intervals D_α. Since now we do not need to eliminate those with $f(D_\alpha) \subseteq D_\alpha$, condition 3 in the definition of E_n can be dropped. We have that $E'_n \subset \mathrm{Forb}(f)$ for $n \geq 2k + 2$, where $E'_n \subset \mathcal{S}_n$ is the set of permutations π for which there exists an i such that

1. for each $i \leq j \leq i + k - 1$, there exists $1 \leq \ell \leq n - 1$ such that $\pi(j) > \pi(\ell) > \pi(j + 1)$ and $\pi(\ell) < \pi(\ell + 1)$,
2. $\pi(i + k) > \pi(i + k + 1)$.

For example, $\pi = 35 \cdots (2k + 1)(2k + 2)(2k)(2k - 2) \cdots 421 \in E'_{2k+2}$. $\square$

We can apply Theorem 4.2 to the sawtooth map Saw_N. In this case, D is the union of $k = N - 1$ intervals, and the theorem guarantees that the shortest forbidden pattern has length at most $2N$. In fact, it is shown in [2] that the shortest forbidden pattern of Saw_N has length $N + 2$. For small values of N, it follows from the proof of Theorem 4.2 that $3421 \in \mathrm{Forb}(\mathrm{Saw}_2)$ and $356\,421 \in \mathrm{Forb}(\mathrm{Saw}_3)$.

Our theorem gives a tight bound when applied to some alternating sawtooth maps. For the map Alt_N where N is odd, we see that D has $k = (N - 1)/2$ components (see Fig. 1). In this case, Theorem 4.2 states that the shortest forbidden pattern of Alt_N has length at most $N + 1$, and this turns out to be its actual length, as shown in [1, Theorem 4.5].

Both Theorems 4.1 and 4.2 have analogous symmetric formulations if we consider the set $\{x \in I : f(x) > x\}$ instead of D. For example, here is the corresponding version of Theorem 4.1.

Corollary 4.3. *Let f be a piecewise monotone map with m monotonicity intervals. Let $k \leq m$ be the number of such intervals I' with one of these two properties:*

- *f is increasing in I' and the right endpoint a of I' satisfies $f(a) > a$,*
- *f is decreasing in I' and I' contains a point with $f(x) > x$.*

Then the length of the shortest forbidden pattern of f is at most $2k + 3$.

5. Antichains that are basic forbidden patterns of a function

In Section 1, we mentioned the problem of characterizing those sets Σ for which there exists a piecewise monotone map f such that $\mathrm{B}(f) = \Sigma$. Aside from the obvious prerequisite that Σ has to be an antichain in P_c, another necessary condition is that the number of permutations avoiding Σ must grow at most exponentially. Using this requirement, we can show that certain finite antichains are not of the form $\mathrm{B}(f)$. In the next proposition, the floor function $[x]$ is the largest integer that is less than or equal to x.

Proposition 5.1. *Let $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ be a finite antichain in P_c and let k_i be the length of σ_i, with $k_1 \leq k_2 \leq \cdots \leq k_m$. If Σ is the set of basic forbidden patterns of a piecewise monotone map, then*

$$k_1 + k_2 + \cdots + k_m \geq [k_1/2]! + m([k_1/2] - 1).$$

Proof. Let $\ell = [k_1/2]$. Assume to the contrary that $k_1 + k_2 + \cdots + k_m < \ell! + m(\ell - 1)$. Equivalently,

$$\sum_{i=1}^{m}(k_i - \ell + 1) < \ell!.$$

There are $k_i - \ell + 1$ consecutive subpatterns (not necessarily different) of length ℓ in σ_i. Thus, the above inequality implies that there is at least one permutation π of length ℓ that is not contained in any of $\sigma_1, \sigma_2, \ldots, \sigma_m$.

Let

$$G_{r\ell} = \{\tau_1\tau_2 \cdots \tau_r \in \mathcal{S}_{r\ell} : \rho(\tau_i) = \pi \text{ for } 1 \leq i \leq r\}.$$

We claim that $G_{r\ell} \subseteq \mathrm{Av}(\Sigma)$. To see this, note that every subpattern of $\tau_1\tau_2 \cdots \tau_r$ of length at least k_1 spans the entirety of some τ_i, so it contains π. On the other hand, no permutation in Σ contains π. Therefore, no permutation in Σ is a subpattern of any $\tau_1\tau_2 \cdots \tau_r \in G_{r\ell}$, so $\tau_1\tau_2 \cdots \tau_r$ avoids Σ.

The size of $G_{r\ell}$ is equal to the number of ways to partition the set $\{1, 2, \ldots, r\ell\}$ into r blocks of size ℓ, which is

$$|G_{r\ell}| = \binom{r\ell}{\ell, \ell, \ldots, \ell} = \frac{(r\ell)!}{(\ell!)^r}.$$

Using Stirling's formula, we see that, as r goes to infinity,

$$|\operatorname{Av}_{r\ell}(\Sigma)| \geq |G_{r\ell}| \gg \frac{(r\ell)^{r\ell}}{\mathrm{e}^{r\ell}(\ell!)^r} = \left(\frac{\ell^\ell}{\mathrm{e}^\ell \ell!}\right)^r r^r,$$

so $|\operatorname{Av}_{r\ell}(\Sigma)|$ grows super-exponentially. Thus Σ cannot be the set of basic forbidden patterns of a piecewise monotone map. □

Theorem 1.1 states that, if σ has length $k \geq 3$, $|\operatorname{Av}_n(\sigma)|$ grows super-exponentially. When $k \geq 6$, this result can be directly derived from the above proposition. To see this, note that, when $k_1 \geq 6$, $[k_1/2] \geq 3$, and so

$$[k_1/2]! + n([k_1/2] - 1) - \sum_{i=1}^{n} k_i = [k_1/2]! + [k_1/2] - 1 - k_1 \geq 2[k_1/2] + [k_1/2] - 1 - k_1 > 0.$$

Interestingly, exponential growth on the number of permutations avoiding an antichain is not the only requirement for it to be the set of basic forbidden patterns of a piecewise monotone function. For example, consider the antichain $\Sigma = \{132, 231\}$. Then, as mentioned in the introduction, $\operatorname{Av}(\Sigma)$ is the set of permutations with no peaks. In other words, permutations in $\operatorname{Av}(\Sigma)$ consist of a decreasing sequence followed by an increasing sequence. It is easy to see that $|\operatorname{Av}_n(\Sigma)| = 2^{n-1}$, since such a permutation is determined by the set of elements other than 1 in the initial decreasing sequence. However, even though the exponential growth condition is satisfied, we have the following result.

Proposition 5.2. *There exists no piecewise monotone map f on a closed interval $I \subset \mathbb{R}$ such that $\mathrm{B}(f) = \{132, 231\}$.*

Proof. Assume to the contrary that there exists such a map f. Let m be the number of monotonicity intervals of f. As in the proof of Theorem 4.1, we have that $\pi = (2m+2)(2m)(2m-2)\cdots 42\,135\cdots(2m-1)(2m+1)(2m+3) \in \operatorname{Forb}(f)$. Since π avoids 132 and 231, it is not possible that $\mathrm{B}(f) = \{132, 231\}$. □

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